

# Black Hole Spectroscopy: Determining Waveforms from 3D Excited Black Holes

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We present the first results for Cauchy nonlinear evolution of 3D, nonaxisymmetric distorted black holes. We focus on the extraction and verification of 3D waveforms determined by numerical relativity. We show that the black hole evolution can be accurately followed through the ringdown period, and comparing with a recently developed perturbative evolution technique, we show that many waveforms in the black hole spectrum of modes, such as  $\ell = 2$  and  $\ell = 4$ , including weakly excited nonaxisymmetric modes ( $m \neq 0$ ), can be accurately evolved and extracted from the full nonlinear numerical evolution. We also identify new physics contained in higher modes, due to nonlinear effects. The implications for simulations related to gravitational wave astronomy are discussed.

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Numerical relativity is the main tool for computing waveforms needed to aid in the detection and interpretation of gravitational wave signals from merging black holes (BH's). Recent and very detailed analysis shows that BH mergers may well be the first events seen, but that information from full 3D numerical simulations will be crucial in enhancing the rate of detection and in extracting astrophysical parameters from the signals [1].

In response to this urgent need for 3D simulations, many groups worldwide have organized an effort to compute waveforms for binary BH mergers. Progress has been slow due to the difficulties of evolving the full Einstein equations, particularly in the BH case. For a review of this difficult problem, see [2]. Notable recent progress includes 3D characteristic evolution, success in apparent horizon boundary conditions (AHBC) [3,4], and extraction of waves from 3D simulations of *axisymmetric* BH's [5]. But, to date there have not been any published attempts to evolve, and extract meaningful waveforms from true 3D distorted BH's, as will be formed from the inspiral and merger of two colliding BH's, using full nonlinear numerical relativity. This ability is crucial for gravitational wave astronomy, and in this Letter we show that it can now be performed with a high degree of accuracy.

We consider here the first detailed, nonlinear evolutions of nonaxisymmetric BH's, using the full machinery of standard 3+1 numerical relativity. We focus on a careful analysis of the 3D waveforms extracted, with an independent check on their accuracy and reliability, building particularly on two recent papers. Ref. [5] showed that one can use a full 3D nonlinear evolution code to study the excitation and ringdown of axisymmetric distorted BH's, including the extraction of the waveforms, such as those created by the head-on collision of two BH's. It was further shown, by comparison to 2D codes, that even very low amplitude waves could be extracted accurately in 3D.

Ref. [6] developed a technique to use perturbation theory to evolve distorted BH data sets as a testbed for numerical relativity, showing how well this technique can work on a series of simulations performed with an axisymmetric distorted BH (2D) code, which allows one to verify the fully nonlinear code, and to identify nonperturbative effects.

In this paper, for the first time we combine full 3D numerical relativity with perturbative techniques to show the extent to which nonaxisymmetric, distorted BH's can be evolved, and waveforms extracted accurately. We find that even very low energy ( $E < 10^{-6}M$ , where  $M$  is the BH mass), *nonaxisymmetric* wave modes can be reliably extracted in full 3D simulations. As emphasized by [1], nonaxisymmetric modes, particularly the  $\ell = m = 2$  mode, may be the dominant signal in the final merger waveform. We show that not only can this mode be accurately extracted (albeit for nonrotating spacetimes at present), but even the much weaker  $\ell = 4, m$  family can be accurately computed. For a BH formed during some violent process (such as collision with another BH), we expect many modes to be excited. The main points of this paper are that (a) entire families of waveforms from the spectrum of an excited BH can now be reliably determined in full 3D numerical relativity, verified by perturbation theory, and (b) nonlinear physics can also be identified. The combination of 3D nonlinear evolution and perturbative treatment is very powerful, and provides an important testing ground for all 3D codes used to study waveforms from BH merger events, that will be crucial for gravitational wave astronomy.

**3D Simulations and Comparison with Perturbative Evolution.** We consider the nonlinear numerical evolution of full 3D distorted BH's with a code written in 3D Cartesian coordinates, described in Refs. [5,7–10]. The code is built on earlier 3D work, where spherical [9]

and colliding BH's [11], and pure waves [10] were studied, and was used in [5,7,8], where axisymmetric distorted BH data sets were studied in 3D. Preliminary attempts to extract waveforms from 3D BH's were made in [7]. This code solves the full set of Einstein equations with general gauge and slicing conditions, but here we use zero shift and the “1+log” algebraic lapse condition [9].

The inner boundary on the BH is provided by an isometry condition [5,7,8]. Ultimately, one expects to replace this with an AHBC [2], preventing the need to bend time slices so much that they ultimately destroy the simulations [9]. The outer boundary conditions keep all functions fixed in time. If the boundary is far enough away this condition is usually adequate, but at present in 3D, without adaptive meshes (AMR) [12,13], it is generally impractical. This can create reflections of the waves at late times. Better boundary conditions, such as the perturbative outer boundary [14], Cauchy-Characteristic matching [15], other treatments provided by hyperbolic formulations of Einstein's equations, or AMR should improve this situation. But these issues do not affect the main point we make below, which is that *high resolution and verifiable studies of truly 3D BH's, including the ringdown and propagation of very low amplitude nonaxisymmetric radiation modes, are now possible.*

The initial data we evolve in this paper belong to the family of distorted BH's discussed in Refs. [5,7,8,16–18], and correspond to a “Brill wave” [19] superimposed on a BH. Such data sets mimic the endstate of two BH's colliding, forming a useful model for studying the late stages of BH mergers. To summarize we simply write the 3-metric here as  $dl^2 = \tilde{\psi}^4 (e^{2q} (d\eta^2 + d\theta^2) + \sin^2 \theta d\phi^2)$ , where  $\eta$  is a radial coordinate related to the Cartesian coordinates by  $\sqrt{x^2 + y^2 + z^2} = e^\eta$  [5]. Given a choice for the “Brill wave” function  $q$ , the Hamiltonian constraint leads to an elliptic equation for the conformal factor  $\tilde{\psi}$ . The function  $q$  represents the wave surrounding the BH, and is chosen to be  $q(\eta, \theta, \phi) = a \sin^n \theta \left( e^{-(\frac{\eta+b}{w})^2} + e^{-(\frac{\eta-b}{w})^2} \right) (1 + c \cos^2 \phi)$ . If the amplitude  $a$  vanishes, the undistorted Schwarzschild solution results; small values of  $a$  correspond to a perturbed BH. Although these data sets can include angular momentum [20], we focus here on the time symmetric case, so only the Hamiltonian constraint need be solved for the conformal factor  $\tilde{\psi}$ . Note that the form of the 3-metric, although fully nonaxisymmetric, does have a discrete (quadrant) symmetry about the  $z$ -axis and the equatorial plane. Hence we can evolve the BH in an octant and save on the memory and computation required.

*Full 3D Nonlinear Evolution.* The initial data set ( $a = -0.1, b = 0, c = 0.5, w = 1, n = 4$ ) was evolved with the 3D code described above with a grid spacing of  $\Delta x = \Delta y = \Delta z = 0.093M$ , with 300 grid zones in each coordinate direction. As in previous spherical and axisymmetric studies, the evolutions can be carried out

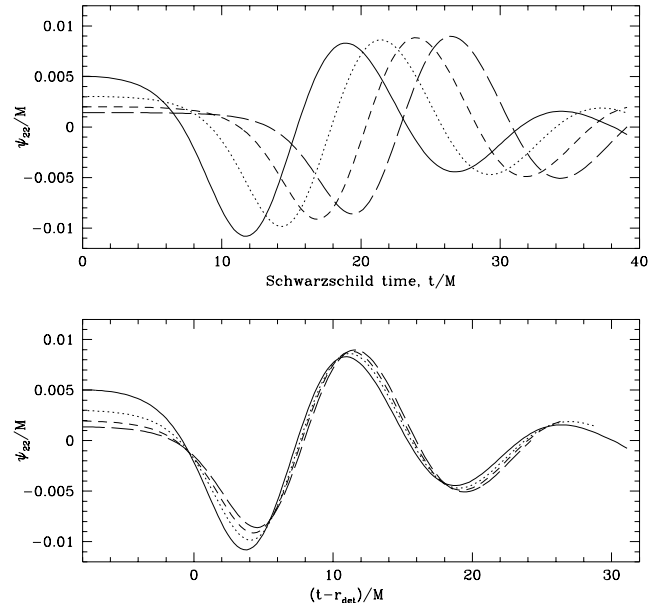


FIG. 1. We show waveforms for the  $\ell = m = 2$  nonaxisymmetric mode extracted from the full nonlinear 3D simulation of the distorted BH. The top graph shows the waveform extracted at different locations, while the bottom shows the phase-shifted waveforms, allowing comparison between them.

through about  $t = 40M$ , when the large gradients in metric functions, created by the use of singularity avoiding time slicings, cause the code to become rather inaccurate and crash. The problems encountered, not relevant for the main results of this work, should be eliminated through the use of AHBC (see, e.g. [2] and references therein).

During the evolution, we extract the waves using a gauge-invariant technique developed originally by Abrahams [21,22]. The details of the present 3D application are given in [6–8]. Essentially, the Zerilli function  $\psi$  is computed at various radii away from the distorted BH. In this work we extracted waveforms at Schwarzschild radii,  $r$ , of  $8.0M$ ,  $10.2M$ ,  $12.6M$  and  $14.9M$ . The 3D code and waveform extraction procedure were developed and verified through careful comparison to an axisymmetric code [16,17] and also on full 3D initial data [6]. Here we apply the same techniques to study the first waveforms emitted from *nonaxisymmetric* distorted BH's in full numerical relativity.

In Fig. 1 we show the nonaxisymmetric  $\ell = m = 2$  mode emitted during the evolution of this distorted BH. Waves are extracted at the different radii, showing the development of this mode in time. The quasinormal ringing of the BH is clearly evident and has the correct complex frequency. The modes are normalized so that the energy they carry is given by  $\dot{E} = (1/32\pi)\dot{\psi}^2$ . For this mode, we find that the energy is about  $10^{-6}M$ .

As mentioned, this 3D mode is expected to be an im-

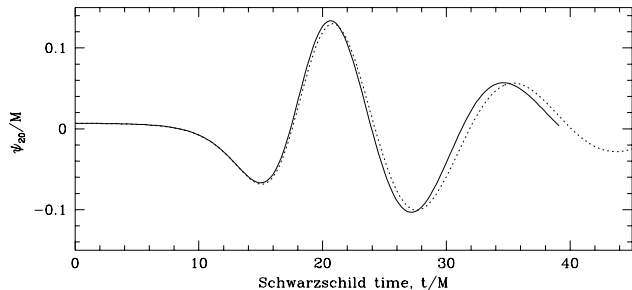


FIG. 2. We show the waveform for the  $\ell = 2, m = 0$  mode, extracted from the linear and nonlinear evolution codes. The dotted (solid) line shows the linear (nonlinear) evolution.

portant and dominant mode in the spectrum of BH excitations created during the merger process. This is the first example showing that nonaxisymmetric modes can be computed in a full 3D simulation, in spite of all the difficulties of BH simulations. The calculation was repeated at higher and lower grid resolutions, the waveform was found to be converging to between first and second order in the grid spacing, with the changes in the waveform at different resolutions being minimal. Beyond resolution studies, having an independent technique to determine the expected waveform, to which comparisons can be made, is also essential in studying their quality. As emphasized elsewhere [23], perturbative techniques can also be used as a testbed to study black hole dynamics, and provide physical insight into the nature of the solution. We turn to perturbation theory in the next section to provide this testbed, and the physical insight it brings.

*Comparison with Perturbative Evolution.* Although we have seen that the 3D modes have the right quasinormal frequency, without a careful check of the simulations, we have no guarantee that the results are actually correct. However, building on prior work of Abrahams, Price, Pullin and others (see [23] for recent review), we have recently developed a technique to study distorted BH's through a perturbative approach to the evolution [6]. The technique makes use of the extraction technique used above to extract initial data for the Zerilli equation, describing linearized perturbations of spherical BH's. As long as the initial wave content on the Schwarzschild background is small, one should be able to use this simple linear wave equation to evolve the gravitational waves and predict the waveforms that must be seen by the full 3D, nonlinear code. We have taken such initial data, evolved it with the Zerilli equation, and compared with waveforms extracted from the full 3D nonlinear evolution in Cartesian coordinates.

In Figs. 2, 3, and 4 we show results for a large part of the BH spectrum of modes excited. The waveforms for the linear and nonlinear evolutions are each plotted on the same graphs, extracted at  $r = 12.6M$ . The total energy radiated for these modes runs from  $E \sim 3 \times 10^{-4}M$

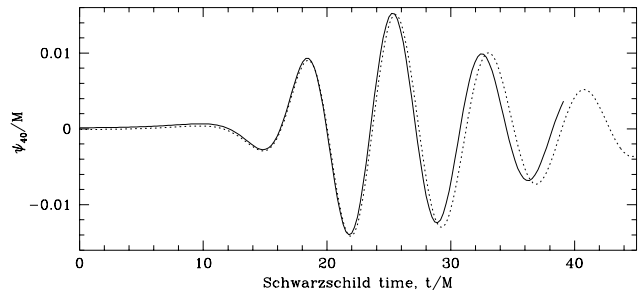


FIG. 3. Waveforms are shown for the  $\ell = 4, m = 0$  extracted from the linear and nonlinear evolution codes. The dotted (solid) line shows the linear (nonlinear) evolution.

for the  $\ell = 2, m = 0$  mode, to  $E \sim 3 \times 10^{-7}M$  for the  $\ell = 4, m = 2$  mode. The agreement between these two completely independent treatments is remarkable, giving complete confidence in the reliability of these waveforms. Not only do the perturbative results confirm those of full 3D numerical relativity, but the 3D results confirm the perturbative treatment. This is an important point to emphasize, as it shows that these modes are actually in the perturbative regime. Hence the gravitational wave physics of these modes is adequately treated by linearized theory. This must be captured by the full 3D simulation, even though the singularity avoiding slicing chosen for the 3D simulation forces the BH background to evolve in a fully nonlinear manner (leading to “grid stretching”, etc.) The  $\ell = m = 2$  waveform already shown above has the same level of agreement with its perturbative counterpart. A full analysis of these waveforms and other 3D BH spacetimes will be published elsewhere [24].

We emphasize that although the amplitude of distortion of this BH is low enough that some modes can be treated with linear perturbation theory, there are limits to its applicability. Other modes can be out of the linear regime, and nonlinear evolution should not agree with linearized evolution results. Detailed analysis of the initial data shows that only the modes  $\ell = 2, m = 0, 2$  and  $\ell = 4, m = 0, 2$  should occur at linear order in the wave amplitude  $a$ , all other modes occurring at order  $a^2$  or above. These higher order modes cannot be accurately evolved with the first order Zerilli equation; Ref. [25] shows that in such cases there is a source term arising from *nonlinear* mixing of the modes that appear at linear order. In some sense, these modes are *too small* for linear treatment to adequately capture the physics, yet as we show now they can be treated in full 3D numerical relativity.

In Fig. 5, we consider such a low amplitude case,  $\ell = 6, m = 2$ . (Other modes are more difficult to extract because of numerical resolution, e.g.  $\ell = m = 4$  has not converged in our simulations. Full details will be published elsewhere). This is a difficult mode to extract numerically in 3D [24], but the waveform is not significantly

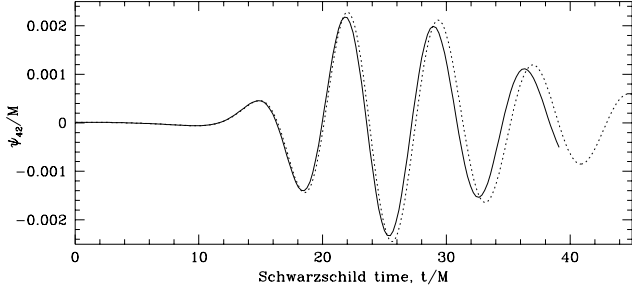


FIG. 4. Waveforms are shown for the  $\ell = 4, m = 2$  mode, extracted from the linear and nonlinear evolution codes. The dotted (solid) line shows the linear (nonlinear) evolution.

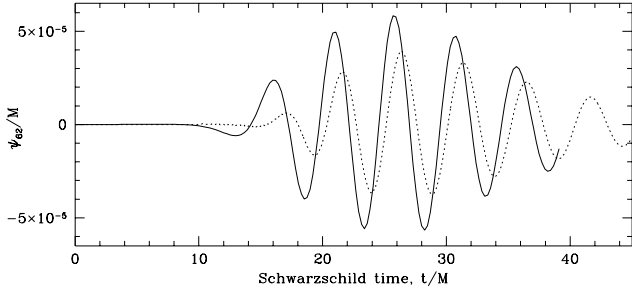


FIG. 5. Waveforms are shown for the  $\ell = 6, m = 2$  mode, extracted from the linear and nonlinear evolution codes. The dotted (solid) line shows the linear (nonlinear) evolution. The discrepancy is attributed to a *nonlinear* effect.

altered by more resolution, and show a marked disagreement in contrast to the previous Figs. The linear energy calculation indicates that the energy of this mode is only  $E \sim 10^{-10}M$ . Simulations at different values of  $a$  show the amplitude of this waveform scaling as  $a^2$ . Hence, we fully expect the perturbative approach to fail for this mode, and we must turn to nonlinear treatments, such as a 3D numerical relativity code, to compute it. This same effect is discussed in [6] for highly resolved 2D cases where nonlinear mode mixing is very clearly seen; details of the more difficult 3D cases are in preparation for publication elsewhere [24].

**Conclusions.** We have shown for the first time that evolutions of a new class of full 3D distorted black holes can be accurately performed in 3D Cauchy nonlinear evolution codes in Cartesian coordinates, and further that a large spectrum of different *nonaxisymmetric* waveforms can now be determined from the simulations, even when they carry a tiny fraction of the ADM mass ( $E \sim 10^{-6}M$ ). Such capability will be crucial in simulating 3D black hole mergers, needed for effective detection of waves. We have also applied newly developed perturbative techniques to verify that the waveforms are very accurate, and also to show *nonlinear* effects. This comparison technique should provide a powerful testbed and analysis tool for all 3D codes under development for use

in gravitational wave astronomy.

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